

# CLASSIFICATION OF $p$ -GROUPS BY THEIR SCHUR MULTIPLIER

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ABSTRACT. Let  $G$  be a non-abelian  $p$ -group of order  $p^n$  and  $M(G)$  be its Schur multiplier. It is well known result by Green that  $|M(G)| \leq p^{\frac{1}{2}n(n-1)}$ . So  $|M(G)| = p^{\frac{1}{2}n(n-1)-t(G)}$  for some  $t(G) \geq 0$ . The groups has already been classified for  $t(G) \leq 5$  by several authors. For  $t(G) = 6$  the classification has been done in [12]. In this paper we classify  $p$ -groups  $G$  for  $t(G) = 6$  in different method.

## 1. INTRODUCTION

The Schur multiplier  $M(G)$  of a group  $G$  was introduced by Schur [1] in 1904 on the study of projective representation of groups.

For  $p$ -groups  $G$  of order  $p^n$ , Green [3] gave an upper bound  $p^{\frac{1}{2}n(n-1)}$  for order of the Schur Multiplier  $M(G)$ . So we have  $|M(G)| = p^{\frac{1}{2}n(n-1)-t(G)}$ , for some  $t(G) \geq 0$ . Now the question comes in our mind that whether it is possible to classify the structure of all  $p$ -groups  $G$  by the order of the Schur multiplier  $M(G)$ , i.e., when  $t(G)$  is known. Several authors have already answered this question. They classified the groups of order  $p^n$  for  $t(G) \leq 5$  in [4, 5, 6, 8, 9]. The structure of  $p$ -groups with  $t(G) = 6$  has been determined in [12]. In the present paper, we classify the structure of all non-abelian finite  $p$ -groups when  $t(G) = 6$ , i.e.  $|M(G)| = p^{\frac{1}{2}n(n-1)-6}$ . Our method is quite different to that of [12]. We have stated some structural results of group  $G$  with the assumption  $t(G) = 6$ .

By  $ES_p(p^3)$  and  $ES_{p^2}(p^3)$  we denote extra-special  $p$ -groups of order  $p^3$  having exponent  $p$  and  $p^2$  respectively. By  $ES_p(p^5)$  and  $ES_{p^2}(p^5)$  we denote extra-special  $p$ -groups of order  $p^5$  having exponent  $p$  and  $p^2$  respectively. By  $\mathbb{Z}_p^{(k)}$  we denote  $\mathbb{Z}_p \times \mathbb{Z}_p \cdots \times \mathbb{Z}_p$  ( $k$  times).

James [19] classified all  $p$ -groups of order  $p^n$  for  $n \leq 6$  upto isoclinism which are denoted by  $\Phi_k$ . We use his notation throughout this paper.

In this paper we prove the following result.

**Theorem 1.1.** (*Main Theorem*) *Let  $G$  be a non-abelian  $p$ -group of order  $p^n$  with  $|M(G)| = p^{\frac{1}{2}n(n-1)-6}$ .*

*If  $p$  is odd, then  $G$  is isomorphic to*

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- (i)  $ES_p(p^3) \times \mathbb{Z}_p^{(5)}$ .
- (ii)  $\Phi_2(21^4)a = ES_{p^2}(p^3) \times \mathbb{Z}_p^{(3)}$ .
- (iii)  $\Phi_2(21^4)b = \Phi_2(211)b \times \mathbb{Z}_p^{(2)}$ ,
- where  $\Phi_2(211)b = \langle \alpha, \alpha_1, \alpha_2, \gamma \mid [\alpha_1, \alpha] = \gamma^p = \alpha_2, \alpha^p = \alpha_1^p = \alpha_2^p = 1 \rangle$ .
- (iv)  $\Phi_5(21^4)a = ES_{p^2}(p^5) \times \mathbb{Z}_p$ .
- (v)  $\Phi_5(1^6) = ES_p(p^5) \times \mathbb{Z}_p$ .
- (vi)  $\Phi_5(21^4)b = \langle \alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta, \gamma \mid [\alpha_1, \alpha_2] = [\alpha_3, \alpha_4] = \gamma^p = \beta, \alpha_i^p = \beta^p = 1 (i = 1, 2, 3, 4) \rangle$ .
- (vii)  $\Phi_4(1^6) = \Phi_4(1^5) \times \mathbb{Z}_p$ ,
- where  $\Phi_4(1^5) = \langle \alpha, \alpha_1, \alpha_2, \beta_1, \beta_2 \mid [\alpha_i, \alpha] = \beta_i, \alpha^p = \alpha_i^p = \beta_i^p = 1 (i = 1, 2) \rangle$ .
- (viii)  $\Phi_2(2111)c = \Phi_2(211)c \times \mathbb{Z}_p$ ,
- where  $\Phi_2(211)c = \langle \alpha, \alpha_1, \alpha_2 \mid [\alpha_1, \alpha] = \alpha_2, \alpha^{p^2} = \alpha_1^p = \alpha_2^p = 1 \rangle$ .
- (ix)  $\Phi_2(2111)d = ES_p(p^3) \times \mathbb{Z}_{p^2}$ .
- (x)  $\Phi_3(1^5) = \Phi_3(1^4) \times \mathbb{Z}_p$ ,
- where  $\Phi_3(1^4) = \langle \alpha, \alpha_1, \alpha_2, \alpha_3 \mid [\alpha_i, \alpha] = \alpha_{i+1}, \alpha^p = \alpha_i^{(p)} = \alpha_3^p = 1 (i = 1, 2) \rangle$ .
- (xi)  $\Phi_7(1^5) = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \beta \mid [\alpha_i, \alpha] = \alpha_{i+1}, [\alpha_1, \beta] = \alpha_3, \alpha^p = \alpha_1^{(p)} = \alpha_{i+1}^p = \beta^p = 1 (i = 1, 2) \rangle$ .
- (xii)  $\Phi_2(31) = \langle \alpha, \alpha_1, \alpha_2 \mid [\alpha_1, \alpha] = \alpha^{p^2} = \alpha_2, \alpha_1^p = \alpha_2^p = 1 \rangle$ .

If  $p = 2$ , then  $G$  is isomorphic to

- (xiii)  $D_8 \times \mathbb{Z}_2^{(4)}$ ,
  - (xiv)  $\langle a, b, c, d, e \mid [d, c] = [e, b] = a^2, a^4 = b^2 = c^2 = d^2 = e^2 = 1 \rangle$ ,
  - (xv)  $\mathbb{Z}_2 \times \langle a, b, c, d \mid [b, c] = [a, d] = b^2, a^2 = b^4 = c^2 = d^2 = 1 \rangle$ ,
  - (xvi)  $\mathbb{Z}_2 \times \langle a, b, c, d \mid b^2 = c^2, [b, c] = [a, d] = b^2, a^2 = b^4 = c^4 = d^2 = 1 \rangle$ ,
  - (xvii)  $\mathbb{Z}_2 \times \mathbb{Z}_2 \times X$  where  $X = \langle a, b, c \mid [b, c] = a^2, [a, b] = [a, c] = 1, a^4 = b^2 = c^2 = 1 \rangle$
  - (xviii)  $Q_8 \times \mathbb{Z}_2^{(3)}$ ,
  - (xix)  $\mathbb{Z}_2^{(4)} \rtimes \mathbb{Z}_2$ ,
  - (xx)  $\langle a, b, c \mid [a, c] = [b, c] = 1, (ba)^2 = (ab)^2, ba^2 = a^2b, a^4 = b^2 = c^2 = 1 \rangle$ ,
  - (xxi)  $QD_{16}$ ,
  - (xxii)  $Q_{16}$ ,
  - (xxiii)  $\langle a, b \mid [a, b] = a^4, a^8 = b^2 = 1 \rangle$ ,
  - (xxiv)  $\langle a, b, c \mid [a, c] = a^2, [b, c] = b^2, a^4 = b^4 = c^2 = 1 \rangle$ ,
- where  $D_n$  denotes Dihedral group of order  $n$ ,  $QD_n$  denotes QuasiDihedral groups of order  $n$  and  $Q_n$  denotes Quaternion group of order  $n$ .

## 2. PRELIMINARIES

In this section we list following results which are used in our proof.

**Theorem 2.1.** (see [15, Theorem 4.1]) *Let  $G$  be a finite group and  $K$  a central subgroup. Set  $A = G/K$ . Then  $|M(G)||G' \cap K|$  divides  $|M(A)||M(K)||A \otimes K|$*

**Proposition 2.2.** (see [14, Proposition 2.4]) *Let  $G$  be a finite nilpotent group of class  $c \geq 2$ . Then  $|\gamma_c(G)||M(G)| \leq |M(G/\gamma_c(G))||G/Z_{c-1}(G) \otimes \gamma_c(G)|$*

**Theorem 2.3.** (see [20, Corollary 4.16]) *Let  $G$  be an extra special  $p$ -group of order  $p^{2n+1}$ .*

- (i) *If  $n \geq 2$ , then  $|M(G)| = p^{2n^2-n-1}$ .*
- (ii) *If  $n = 1$ , then the Schur multiplier of  $D_8, Q_8, ES_p(p^3)$  and  $ES_{p^2}(p^3)$  are isomorphic to  $\mathbb{Z}_2, 1, \mathbb{Z}_p \times \mathbb{Z}_p$  and 1 respectively.*

The following result follows from [17] for  $|G'| = p$  and from [6, page. 4177] for  $|G'| = p^2$ .

**Theorem 2.4.** *For non-abelian  $p$ -groups  $G$  of order  $p^4$  with  $|G'| = p$ ,  $M(\Phi_2(211)a) \cong \mathbb{Z}_p \times \mathbb{Z}_p$ ,  $M(\Phi_2(1^4)) \cong \mathbb{Z}_p^{(4)}$ ,  $M(\phi_2(31)) \cong 1$ ,  $M(\Phi_2(22)) \cong \mathbb{Z}_p$ ,  $M(\Phi_2(211)b) \cong \mathbb{Z}_p \times \mathbb{Z}_p$ ,  $M(\Phi_2(211)c) \cong \mathbb{Z}_p \times \mathbb{Z}_p$  and For  $|G'| = p^2$ ,  $M(\Phi_3(211)a) \cong \mathbb{Z}_p$ ,  $M(\Phi_3(211)b_r) \cong \mathbb{Z}_p$ ,  $M(\Phi_3(1^4)) \cong \mathbb{Z}_p \times \mathbb{Z}_p$*

The following three lemmas follow from [7, Main theorem].

**Lemma 2.5.** *Let  $G$  be a non-abelian  $p$ -group of order  $p^n$  with  $|M(G)| = p^{\frac{1}{2}n(n-1)-6}$ . Then  $n \leq 8$ .*

**Lemma 2.6.** *Let  $G$  be a non-abelian  $p$ -group of order  $p^8$  with  $|M(G)| = p^{\frac{1}{2}n(n-1)-6}$ . Then  $G \cong ES_p(p^3) \times \mathbb{Z}_p^{(5)}$ .*

**Lemma 2.7.** *There is no  $p$ -group  $G$  of order  $p^n$  ( $n \geq 6$ ) with  $|G'| \geq p^3$  and  $|M(G)| = p^{\frac{1}{2}n(n-1)-6}$ .*

**Lemma 2.8.** *There is no non-abelian  $p$ -group  $G$  of order  $p^7$  with  $|M(G)| = p^{\frac{1}{2}7(7-1)-6} = p^{15}$ .*

*Proof.* It follows from [10, Theorem 21]. □

**Lemma 2.9.** *Let  $G$  be a non-abelian  $p$ -group of order  $p^4$  with  $|M(G)| = p^{\frac{1}{2}4(4-1)-6} = 1$ . Then  $G \cong \Phi_2(31)$ .*

*Proof.* This result follows from Theorem 2.4. □

Hence from the discussion above it is clear that we have to study groups of order  $p^5$  and  $p^6$ .

### 3. GROUPS OF ORDER $p^5$

In this section we characterize groups of order  $p^5$  which have Schur multiplier of order  $p^4$ .

**Lemma 3.1.** *There is no non-abelian  $p$ -group of order  $p^5$  with  $|G'| = p^3$  and  $|M(G)| = p^{\frac{1}{2}5(5-1)-6} = p^4$ .*

*Proof.* Notice that nilpotency class of  $G$  is either 3 or 4.

Let  $G$  be of class 4. Then  $G$  lie in the isoclinism classes  $\Phi_9$  or  $\Phi_{10}$ . By Proposition 2.2 and Theorem 2.4 we have  $|M(G)| \leq p^3$ .

Now assume that the nilpotency class of  $G$  is 3. Then it follows that  $G$  lie in  $\Phi_6$  in [19]. In this case take any subgroup  $K \subset Z(G) \cap G'$  of order  $p$ . By Theorem 2.1 we have  $|M(G)|p \leq |M(G/K)|p^2$ . Here  $G/K$  is of order  $p^4$  with  $|(G/K)'| = p^2$ . Hence it follows from Theorem 2.4 that  $|M(G)| \leq p^3$ . This concludes the proof.  $\square$

Before we proceed to the next result, we explain a method by Blackburn and Evens [16] for computing Schur multiplier of  $p$ -groups of class 2 with  $G/G'$  is elementary abelian.

Here  $G/G'$  and  $G'$  are elementary abelian of order  $p^3$  and  $p^2$  respectively. We can consider  $G/G'$  and  $G'$  as vector spaces over  $GF(p)$ , denote by  $V, W$  respectively. Let  $v_1, v_2 \in V$  such that  $v_i = g_i G', i \in \{1, 2\}$  and take  $(v_1, v_2) = [g_1, g_2]$ .

Let  $X_1$  be the subspace of  $V \otimes W$  spanned by all

$$v_1 \otimes (v_2, v_3) + v_2 \otimes (v_3, v_1) + v_3 \otimes (v_1, v_2)$$

Consider a map  $f : V \rightarrow W$  given by  $f(gG') = g^p, g \in G$ . We denote by  $X_2$  the subspace spanned by all  $v \otimes f(v)$  for  $v \in V$  and take  $X = X_1 + X_2$ . Now consider a homomorphism  $\sigma : V \wedge V \rightarrow V \otimes W/X$  given by

$\sigma(v_1 \wedge v_2) = v_1 \otimes f(v_2) + \binom{p}{2} v_2 \otimes (v_1, v_2) + X$ . Then there exists an abelian group  $M^*$  having a subgroup  $N$  for which

$$1 \rightarrow V \otimes W/X \rightarrow M^* \xrightarrow{\xi} V \wedge V \rightarrow 1$$

is exact, where  $N \cong V \otimes W/X, M^*/N \cong V \wedge V$  and  $(\sigma\xi)(m) = m^p$  for all  $m \in M^*$ .

**Theorem 3.2.** [16] *With the notation above, consider a homomorphism  $\rho : V \wedge V \rightarrow W$  given by*

$$\rho(v_1 \wedge v_2) = (v_1, v_2) \text{ for all } v_1, v_2 \in V.$$

*Denote by  $M$ , the subgroup of  $M^*$  containing  $N$  for which  $M/N$  corresponds to  $\ker \rho$ . Then  $M(G) \cong M$*

**Lemma 3.3.** *Let  $G$  be a non-abelian  $p$ -group of order  $p^5$  with  $|G'| = p^2$  and  $|M(G)| = p^{\frac{1}{2}5(5-1)-6} = p^4$ . Then  $G \cong \Phi_3(1^5), \Phi_7(1^5)$ .*

*Proof.* Groups of order  $p^5$  with  $|G'| = p^2$  lie in the isoclinism classes  $\Phi_3, \Phi_4, \Phi_7$  or  $\Phi_8$  [19].

In isoclinism class  $\Phi_3$ , some groups are direct product of its subgroups. So for them we can easily compute  $M(G)$  and we see  $|M(\Phi_3(1^5))| = p^4$ . Now we consider other groups. We have  $|M(G)|p \leq |M(G/\gamma_3(G))|p^2$  using Proposition 2.2. Observe that  $|M(G/\gamma_3(G))| \leq p^2$  except  $\Phi_3(2111)c$ . Therefore  $|M(G)| \leq p^3$  except

$\Phi_3(2111)c$ . For  $G \cong \Phi_3(2111)c$  by Theorem 2.1 (taking  $K = Z(G)$ ) it follows that  $|M(G)| \leq p^3$ .

In class  $\Phi_4$ , every group  $G$  has nilpotency class 2 with  $G/G'$  elementary abelian. By Theorem 3.2  $|M(G)|/|N| = |V \wedge V|/|W|$  and thus one can show that  $G = \Phi_4(1^5)$  has  $|M(G)| = p^6$  but for all other groups  $G$  in  $\Phi_4$ ,  $|M(G)| \leq p^3$ .

In class  $\Phi_7$ , for the groups  $G \cong \Phi_7(2111)a, \Phi_7(2111)b_r, \Phi_7(2111)c$  we can choose a normal subgroup  $K$  such that  $G/K$  is cyclic with  $|M(K)| = p, |K'| = p^2$ . Hence  $|M(G)| \leq p^3$  by [14, Theorem 3.1]. Now by [18, Theorem 2.3.10] we have the following sequence is exact.

$$\text{Hom}(Z, \mathbb{C}^*) \xrightarrow{\text{Tra}} M(G/Z) \xrightarrow{\text{Inf}} M(G) \xrightarrow{\delta} G/G' \otimes Z$$

In particular for  $G \cong \Phi_7(1^5)$ , taking  $Z = Z(G)$ ,  $\text{Im}(\text{Tra}) \cong \ker(\text{Inf}) \cong G' \cap Z \cong \mathbb{Z}_p$ . We can see in [19],  $G$  is capable as  $E/Z(E) \cong G$  for groups  $E$  of order  $p^6$  in isoclinism class  $\Phi_{30}$ . Thus it follows from [18, Corollary 2.5.8 and 2.5.10] that  $\delta$  is not trivial map. So  $|\ker(\delta)| = |\text{Im}(\text{Inf})| = \frac{|M(G/Z)|}{p} = p^3 < |M(G)|$ . So  $p^4 \leq |M(G)|$  and by [14, Theorem 3.1] (taking  $K = \langle \alpha, \alpha_1, \alpha_2, \alpha_3 \rangle$ )  $|M(G)| \leq p^4$ . Hence we conclude  $|M(\Phi_7(1^5))| = p^4$ .

Finally consider the class  $\Phi_8$ , consisting of only one group  $\Phi_8(32)$ . Observe that  $|M(\Phi_8(32))| \leq p^3$  by Theorem 2.1 (taking  $K = Z(G)$ ).  $\square$

**Lemma 3.4.** *Let  $G$  be a non-abelian  $p$ -group of order  $p^5$  with  $|G'| = p$  and  $|M(G)| = p^{\frac{1}{2}5(5-1)-6} = p^4$ . Then  $G \cong \Phi_2(2111)c, \Phi_2(2111)d$ .*

*Proof.* Groups of order  $p^5$  with  $|G'| = p$  lie in the isoclinism classes  $\Phi_2$  and  $\Phi_5$  [19]. The isoclinism class  $\Phi_5$  consists of extra-special  $p$ -groups which have Schur multiplier of order  $p^5$  by Theorem 2.3. Now we consider the class  $\Phi_2$ . Since  $|M(G)| = p^4$ , by Theorem 2.1 we have  $G/G' \cong \mathbb{Z}_{p^2} \times \mathbb{Z}_p \times \mathbb{Z}_p$  or  $\mathbb{Z}_p^{(4)}$ . Now it follows that  $G$  is either isomorphic to  $\Phi_2(311)b, \Phi_2(221)c$  or  $G$  is direct product of its subgroups.

If  $G \cong \Phi_2(311)b, \Phi_2(221)c$ , then by Theorem 2.1  $|M(G)| \leq p^3$  for suitable  $K$ . If  $G$  is direct product of its subgroups, then  $|M(G)| = p^4$  only for  $\Phi_2(2111)c, \Phi_2(2111)d$ .  $\square$

#### 4. GROUPS OF ORDER $p^6$ AND PROOF OF MAIN THEOREM

In this section we characterize groups of order  $p^6$  having Schur multiplier of order  $p^9$  and prove the Main Theorem.

**Lemma 4.1.** *Let  $G$  be a non-abelian  $p$ -group of order  $p^6$  with  $|M(G)| = p^{\frac{1}{2}6(6-1)-6} = p^9$ . Then  $G/G'$  is elementary abelian.*

*Proof.* From Lemma 2.7 we have  $|G'| \leq p^2$ . It follows from [19] that if  $|G'| = p$ , then  $G/Z(G)$  is isomorphic to  $\mathbb{Z}_p \times \mathbb{Z}_p$  or  $\mathbb{Z}_p^{(4)}$ . If  $G' \cong \mathbb{Z}_p \times \mathbb{Z}_p$  then  $G/Z(G)$  is isomorphic

to  $ES_p(p^3), \mathbb{Z}_p^{(3)}, ES_p(p^3) \times \mathbb{Z}_p, ES_p(p^3) \times \mathbb{Z}_p^{(2)}$  or  $\mathbb{Z}_p^{(4)}$ . If  $G' \cong \mathbb{Z}_{p^2}$ , then again by [19]  $G/Z(G)$  is isomorphic to  $\Phi_2(22)$  or  $\mathbb{Z}_{p^2} \times \mathbb{Z}_{p^2}$ . In all the above cases using [11, Proposition 1] we conclude that if  $G/G'$  is not elementary abelian, then  $|M(G)| < p^9$ , which is not our case. Hence the result follows.  $\square$

**Lemma 4.2.** *Let  $G$  be a non-abelian  $p$ -group of order  $p^6$  with  $|G'| = p$  and  $|M(G)| = p^9$ , then  $G$  is isomorphic to one of the following:  $\Phi_2(21^4)a, \Phi_2(21^4)b, \Phi_5(21^4)a, \Phi_5(1^6), \Phi_5(21^4)b$ .*

*Proof.* It follows from [19] that  $G$  lie in the isoclinism classes  $\Phi_2$  or  $\Phi_5$ . By the preceeding lemma we have  $G/G'$  is elementary abelian of order  $p^5$ . So in the isoclinism class  $\Phi_2$  we have to check  $\Phi_2(21^4)a, \Phi_2(21^4)b, \Phi_2(21^4)c, \Phi_2(21^4)d, \Phi_2(1^6)$  and in the isoclinism class  $\Phi_5$  we have to check  $\Phi_5(21^4)a, \Phi_5(1^6), \Phi_5(21^4)b$ .

If  $G \cong \Phi_5(21^4)b$ , then by Theorem 2.1 (taking  $K = Z(G)$ ) we have  $|M(G)| \leq p^9$  and by [15, Corollary 3.2]  $|M(G)| \geq p^9$ . Hence  $\Phi_5(21^4)b$  has Schur multiplier of order  $p^9$ .

All other above groups are direct product of its subgroups. So it is easy to see that among them  $\Phi_2(21^4)a, \Phi_2(21^4)b, \Phi_5(21^4)a, \Phi_5(1^6)$  have Schur multiplier of order  $p^9$ .  $\square$

**Lemma 4.3.** *There is no non-abelian  $p$ -group of order  $p^6$  with  $G' \cong \mathbb{Z}_{p^2}$  and  $|M(G)| = p^9$ .*

*Proof.* From [19] it suffices to consider isoclinism classes  $\Phi_8, \Phi_{14}$ . Now using Theorem 2.1 (taking  $K = Z(G)$ ) we can easily show that  $|M(G)| < p^9$ .  $\square$

**Lemma 4.4.** *Let  $G$  be a non-abelian  $p$ -group of order  $p^6$  with  $G' \cong \mathbb{Z}_p \times \mathbb{Z}_p$  and  $|M(G)| = p^9$ . Then  $Z(G)$  is of exponent  $p$  and  $G' \subseteq Z(G)$ .*

*Proof.* Let the exponent of  $Z(G)$  be  $p^k$  ( $k \geq 2$ ) and  $K$  be a cyclic central subgroup of order  $p^k$ . Then using Theorem 2.1 and [7], we have

$$|M(G)| \leq p^{-1}|M(G/K)||G/K \otimes K| \leq p^{-1}p^{\frac{1}{2}(n-3)(n-4)+1}p^{(n-3)}.$$

When  $n = 6$  it gives  $|M(G)| < p^9$ , which is a contradiction. Hence  $Z(G)$  is of exponent  $p$ .

Now assume that  $G' \not\subseteq Z(G)$ . Then it follows that  $|Z(G)| \leq p^3$ . If  $|Z(G)| = p$ , then  $G$  is of nilpotency class 3. Now by Proposition 2.2, we get  $|M(G)| < p^9$ . If  $|Z(G)| = p^2$ , then by the assumption there is a central subgroup  $K$  of order  $p$  such that  $G' \cap K = 1$  and  $(G/K)' = p^2$ . By Theorem 2.1,  $|M(G)| \leq |M(G/K)|p^3$ . This is possible only when  $|M(G/K)| \geq p^6$ . Now by Theorem [7, 8] we see that there is no such  $G/K$  of order  $p^5$  such that  $|M(G/K)| \geq p^6$  with  $(G/K)' = p^2$  and  $Z(G/K) = p$ . Finally if  $|Z(G)| = p^3$ , then consider a central subgroup of order  $p^2$  such that  $G' \cap K = 1$ . Then  $|M(G)| \leq |M(G/K)|p^4$  by Theorem 2.1. Therefore it follows from Theorem 2.4  $|M(G)| < p^9$ .  $\square$

**Lemma 4.5.** *Let  $G$  be a non-abelian  $p$ -group of order  $p^6$  with  $G' \cong \mathbb{Z}_p \times \mathbb{Z}_p$  and  $|M(G)| = p^{\frac{1}{2}6(6-1)-6} = p^9$ . Then  $G$  is isomorphic to  $\Phi_4(1^6)$ .*

*Proof.* By preceding lemma,  $G' \subseteq Z(G)$ . Now consider a central subgroup  $K$  in  $G'$  of order  $p$ . By Theorem 2.1 we have  $|M(G)|p \leq |M(G/K)|p^4$ . This is possible only when  $|M(G/K)| \geq p^6$ . Since  $(G/K)' \cong \mathbb{Z}_p$  so  $G/K \cong ES_p(p^3) \times \mathbb{Z}_p^{(2)}$  by [7]. This tells that  $G$  is of exponent  $p$ . So we have to study the groups of exponent  $p$  in the isoclinism classes  $\Phi_3, \Phi_4, \Phi_7, \Phi_{22}$  which are  $\Phi_3(1^6), \Phi_4(1^6), \Phi_7(1^6), \Phi_{22}(1^6)$ .

If  $G \cong \Phi_{22}(1^6)$ , then by [14, Theorem 3.1] (taking  $K = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \beta_1 \rangle$ ) we have  $|M(G)| < p^9$ . Other groups are direct product of its subgroups. Hence it is easy to see that only  $\Phi_4(1^6)$  has Schur multiplier of order  $p^9$ .  $\square$

We are now ready to prove our Main Theorem.

**4.1. Proof of Main Theorem.** Let  $G$  be a group of order  $p^n$  ( $p$  odd) with  $|M(G)| = p^{\frac{1}{2}n(n-1)-6}$ . By Lemma 2.5, it follows that  $n \leq 8$ . If  $|G| = p^8$ , then the assertion (i) of the main theorem follows from Lemma 2.6. By Lemma 2.8 it follows that there is no non-abelian  $p$ -group of order  $p^7$  with  $|M(G)| = p^{\frac{1}{2}7(7-1)-6} = p^{15}$ . So the problem reduces to studying groups of order  $p^4, p^5$  and  $p^6$ . If  $|G| = p^4$ , then the assertion (xii) follows from Lemma 2.9. Now we assume  $|G| = p^5$ . Then assertions (viii), (ix) follows from Lemma 3.4 and (x), (xi) follows from Lemma 3.3. Now consider groups of order  $p^6$ . Then the assertions (ii), (iii), (iv), (v), (vi) follow from Lemma 4.2 and (vii) follows from 4.5.

For the case  $p = 2$ , the classification follows from computation using HAP package [21] of GAP [22].  $\square$

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